MID-SEMESTER EXAM ISI BANGALORE MARCH 2014

INTRODUCTION TO DIFFERENTIAL GEOMETRY 100 POINTS

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{R}$ = real numbers.

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1. [12 points] Let $F(x, y, z) = x + y - 2xz$, $G(x, y, z) = y + z + x^2y + 2xz$. Find constants $c, d \in \mathbb{R}$ and a point $p \in \mathbb{R}^3$ such that the intersection of the level surfaces $F = c$ and $G = d$ is a regular curve around p with tangent line at p parallel to $(1, 1, 1)$.

2. [16 points]

Let $\alpha(t)$ be a regular curve in $\mathbb{R}^2$ and let $p = \alpha(a)$ be a point of the curve. Prove that there is a rigid motion (isometry) $M = A \circ L$ in $\mathbb{R}^2$ (with L a translation and A a linear map) and a reparametrization $\beta(\tau)$ of the curve $M\alpha(t)$ around Mp such that $\beta(\tau) = (\tau, f(\tau))$ for a suitable C^∞ function f and the following hold:

$$Mp = \beta(0) = (0, 0), \qquad A\alpha'(0) = \beta'(0) = (1, 0), \qquad f''(0) \geq 0.$$

3. [13 points] Let $\alpha(t)$ be a regular curve on a sphere of radius R in $\mathbb{R}^3$. Prove that $\kappa(t) \geq 1/R$. (Hint: Differentiate $\alpha \cdot \alpha = R^2$ twice.)

4. [21 points] Suppose $B(s)\colon I \to \mathbb{R}^3$ is a C^∞ function such that for all $s \in I$, $||B(s)|| = 1$ and $B, \dot{B}, \ddot{B}$ are linearly independent. Prove that there is always a regular curve $\alpha(s)$ whose binormal vector is $B(s)$. Is the curve uniquely determined ?

5. [12 points] Let $U = \mathbb{R}^2 \setminus (0, 0)$. Consider the vector field $X(x_1, x_2) = (-x_1, -x_2)/\sqrt{x_1^2 + x_2^2}$. For any $p = (a, b) \in U$, write down the maximal integral curve $\alpha(t)$ to X such that $\alpha(0) = p$. (You must specify the domain of α and explain why it is maximal.)

6. [26 points]

Let $U \subset \mathbb{R}^3$ be an open set. Let $f(x, y, z)$ be a C^∞ function on U.

(i) For any C^∞ vector field X on U, how is Xf defined ?

(ii) If X, Y are constant vector fields on U, prove that $XYf = YXf$.

(iii) Give a counterexample to show that the conclusion of (ii) doesn't hold if X is not assumed to be constant.

(Hint: Write X, Y as linear combinations of the standard Euclidean frame field)